CONTROLLABILITY REGION FOR SYSTEMS WITH TWO AND THREE CONSTRAINTS ON THE CONTROL

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The problem is solved of defining in the phase space a set of initial states from which a linear stationary system can be brought to the origin. The case is considered when the magnitude, linear momentum, and energy of the control are simultaneously constrained, as well as the case when its linear momentum and energy are simultaneously constrained.

1. Statement of the problem. We consider a controllable system described by a linear matrix differential equation with real constant coefficients

$$dx/dt = Ax + Bu \tag{1.1}$$

Here $x = \| x_i \|$, $A = \| a_{ij} \|$, $B = \| b_{is} \|$, $u = \| u_s \|$ are matrices of order $(n \times 1)$, $(n \times n)$, $(n \times r)$, $(r \times 1)$, respectively. By b_s we denote the s th column of matrix B ($b_s \neq 0$ for all $s = 1, \ldots, r$). As admissible controls we take measurable functions $u_s(t)$ $(s = 1, \ldots, r)$, satisfying simultaneously the three inequalities

$$|u_s(t)| \leqslant M_s \qquad (M_s = \text{const} > 0) \tag{1.2}$$

$$\int_0^\infty |u_s(\tau)|\, d\tau \leqslant N_s \qquad (N_s = \text{const} > 0) \tag{1.3}$$

$$\int_0^\infty u_s^2(\tau)\, d\tau \leqslant P_s \qquad (P_s = \text{const} > 0) \tag{1.4}$$

From the physical point of view conditions (1.2), (1.3) and (1.4) specify the boundedness of the magnitude, the linear momentum and the energy of the control, respectively. The general solution of system (1.1) has the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau \tag{1.5}$$

where x_0 is the initial state vector.

We pose the problem of defining in the phase space X a set Q (the region of controllability) of states x_0 for each of which there exists an admissible control bringing the system to the origin. The problem of determining the controllability region Q is considered also for the case when the admissible controls are functions satisfying simultaneously only the two integral constraints (1.3), (1.4).

In [1-3] the problem we have posed was solved with $r = 1$ for the cases when controls satisfying conditions (1.2) or (1.3), (1.4) were admissible. The problem was solved in [4, 5] for the case when constraints (1.2) and (1.3) were imposed simultaneously on the control, and in [6] for the case when constraints (1.2) and (1.4) were imposed.

Let us assume that under a certain admissible control the equality $x(t) = 0$ holds for $t = T$ then from (1.5) we have

$$-x_0 = \int_0^T e^{-A\tau} Bu(\tau)\, d\tau = \sum_{s=1}^{r} \int_0^T e^{-A\tau} b_s u_s(\tau)\, d\tau \tag{1.6}$$

The admissible control under which equality (1.6) is realized, satisfies the conditions

$$\int_0^T |u_s(\tau)|\, d\tau \leqslant N_s \tag{1.7}$$

$$\int_0^T u_s^2(\tau)\, d\tau \leqslant P_s \tag{1.8}$$

The set of controls $u_s(t)$ simultaneously satisfying inequalities (1.2), (1.7) and (1.8) is denoted $\Omega_s^1(T)$, while the one for which the controls satisfy simultaneously the inequalities (1.7) and (1.8) is denoted $\Omega_s^2(T)$. The set of vector-valued functions $u(t)$ such that $u_s(t) \in \Omega_s^m(T)$ $(m = 1, 2)$, is denoted $\Omega^m(T)$. The desired controllability regions are denoted Q^1 and Q^2, respectively. The problem posed can be restated as follows: determine the set Q^m of vectors x_0 for each of which there exists T such that equality (1.6) can be ensured by means of a function $u(t) \in \Omega^m(T)$ $(m = 1, 2)$.

2. Regions of attainability. We introduce the notation

$$v_s(T) = \int_0^T e^{-A\tau} b_s u_s(\tau)\, d\tau, \qquad v(T) = \sum_{s=1} v_s(T) = \int_0^T e^{-A\tau} Bu(\tau)\, d\tau \tag{2.1}$$

and in the space X we consider the attainability regions

$$Q_s^m(T) = \{v_s(T): u_s(t) \in \Omega_s^m(T)\}$$

$$Q^m(T) = \sum_{s=1}^{r} Q_s^m(T) = \{v(T): u(t) \in \Omega^m(T)\}$$

The attainability regions $Q_s^m(T)$ $(s = 1, \ldots, r;\ m = 1, 2)$ and $Q^m(T)$ possess the following properties: 1. Closedness. 2. Convexity. 3. $Q_s^m(T)$ "grows" with the increase of T, i.e., $Q_s^m(T_1) \subset Q_s^m(T_2)$ if $T_1 \leqslant T_2$. 4. Symmetry about the origin.

By using the weak compactness in itself of a sphere in the space $L_2[0, T]$ [7], we can prove the weak compactness in themselves of the sets $\Omega_s^m(T)$ $(m = 1, 2)$. Property 1 follows from the fact that the set $Q_s^m(T)$ is a linear mapping of the set $\Omega_s^m(T)$. Properties 2, 3, 4 follow easily from [2-6, 8, 9].

The relations $Q_s^1(T) \subset Q_s^2(T)$, $Q_s^1 \subset Q_s^2$ and $Q^1 \subset Q^2$ hold because $\Omega_s^1(T) \subset \Omega_s^2(T)$. From the definition of the set $Q_s^m(T)$ it follows that the system

$$dx/dt = Ax + b_s u_s \tag{2.2}$$

can be brought to the origin in time T if and only if its initial state $x_0 \in Q_s^m(T)$. In view of Property 3 the controllability region Q_s^m of system (2.2) is a set of points of space X, which includes $Q_s^m(T)$ as $T \to \infty$. The controllability region Q^m of system (1.1) is obtained as the algebraic sum of the regions Q_s^m $(s = 1, \ldots, r)$:

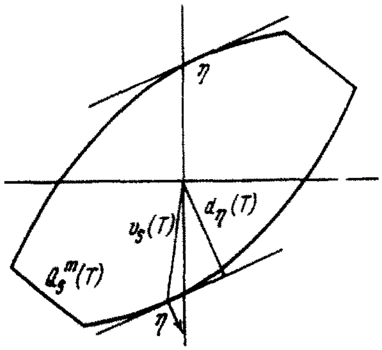

Fig. 1.

$$Q^m = \sum_{s=1}^{r} Q_s^m$$

therefore, we shall first consider the problem of constructing the controllability region Q_s^m of system (2.2).

We take an arbitrary unit $(1 \times n)$-vector η and we construct the support hyperplanes of set $Q_s^m(T)$, orthogonal to vector η. From Properties 2 and 4 it follows that there are two such planes and they are symmetric to each other relative to the origin (Fig. 1). The distance $d_\eta(T)$ from the origin to these planes is given by the expression [10]

$$d_\eta(T) = \max_{v_s(T)\in Q_s^m(T)} (\eta v_s(T)) = \max_{u_s(t)\in \Omega_s^m(T)} \int_0^T \eta e^{-A\tau} b_s u_s(\tau)\, d\tau$$

From Properties 1 and 2 it follows that to the set $Q_s^m(T)$ belong those and only those points x whose coordinates satisfy the inequality

$$|\eta x| \leqslant d_\eta(T)$$

for all possible unit vectors η.

3. Determination of distance $d_\eta(T)$ for the control class $\Omega_s^1(T)$. We solve the problem of maximizing the functional

$$I_s(u) = \int_0^T \eta e^{-A\tau} b_s u_s(\tau)\, d\tau \tag{3.1}$$

with the controls $u_s(t) \in \Omega_s^1(T)$. If $\eta e^{-At} b_s \equiv \mathrm{const}$, then for sufficiently large values of T the maximizing function both in the class $\Omega_s^1(T)$ as well as in the class $\Omega_s^2(T)$ is, obviously, the function $u_s(t) = N_s T^{-1} \operatorname{sgn}(\eta e^{-At} b_s)$ which turns, of the three relations (1.2), (1.7), (1.8), only the relation (1.7) into an equality. Further, we take it that $\eta e^{-At} b_s \not\equiv \mathrm{const}$.

Let $u_s(t)$ be the control solving the problem of maximizing integral (3.1) under the constraints (1.2) and (1.7). It follows from [4] that when $T \geqslant N_s/M_s$ the control $u_s(t)$ equals M_s on some set of measure N_s/M_s and equals zero on the complement of this set with respect to the whole segment $[0, T]$. The integral

$$\int_0^T u_s^2(\tau)\, d\tau$$

after this control has been substituted in it, yields for $T \geqslant N_s/M_s$ an expression for $M_s N_s$. If $M_s N_s \leqslant P_s$, then the control $u_s(t) \in \Omega_s^1(T)$. Consequently, under the condition $M_s N_s \leqslant P_s$ the problem of maximizing integral (3.1) in the control class $\Omega_s^1(T)$ is reduced to the maximizing problem in the presence of only the two conditions (1.2) and (1.7) which was examined in [4].

In what follows we assume that

$$M_sN_s > P_s \tag{3.2}$$

Now let $u_s(t)$ be the control maximizing functional (3.1) under constraints (1.2), (1.8). Two cases are possible: the control $u_s(t)$ does not satisfy inequality (1.7) (Case A), the control $u_s(t)$ satisfies inequality (1.7) (Case B). We first consider Case A. In order to solve the problem of maximizing integral (3.1) we consider the auxiliary functional

$$I_s(u, \chi, \sigma) = \int_0^T \left[\eta e^{-A\tau} b_s u_s(\tau) - \chi\,|u_s(\tau)| - \frac{\sigma}{2} u_s^2(\tau)\right] d\tau \tag{3.3}$$

Here $\chi > 0, \sigma > 0$ are constant Lagrange multipliers. In order to maximize integral (3.3) under condition (1.2) we need to find a function $|u_s(t)| \leqslant M_s$ which maximizes the integrand. Obviously, such a function has the form

$$u_s(t, \chi, \sigma) = \begin{cases} M_s \operatorname{sgn}(\eta e^{-At} b_s), & t \in E_s(T, \chi, \sigma) \\ \sigma^{-1}[|\eta e^{-At} b_s| - \chi] \operatorname{sgn}(\eta e^{-At} b_s), & t \in F_s(T, \chi, \sigma) \\ 0, & t \in G_s(T, \chi) \end{cases} \tag{3.4}$$

$$\begin{gathered} E_s(T, \chi, \sigma) = \{t \in [0, T] : |\eta e^{-At} b_s| > \chi + \sigma M_s\} \\ F_s(T, \chi, \sigma) = \{t \in [0, T] : \chi \leqslant |\eta e^{-At} b_s| \leqslant \chi + \sigma M_s\} \\ G_s(T, \chi) = \{t \in [0, T] : |\eta e^{-At} b_s| \leqslant \chi\} \\ (E_s(T, \chi, \sigma) + F_s(T, \chi, \sigma) + G_s(T, \chi) = [0, T]) \end{gathered} \tag{3.5}$$

We substitute function (3.4) into relations (1.7) and (1.8) and we show that in Case A there exist values of $\chi > 0$ and $\sigma > 0$ for which these relations turn into equalities. After the substitution we obtain the following equations in the variables χ, σ:

$$\Phi_1(\chi, \sigma) = M_s \mu E_s(T, \chi, \sigma) + \frac{1}{\sigma} \int_{F_s(T, \chi, \sigma)} [|\eta e^{-A\tau} b_s| - \chi]\, d\tau = N_s \tag{3.6}$$

$$\Phi_2(\chi, \sigma) = M_s^2 \mu E_s(T, \chi, \sigma) + \frac{1}{\sigma^2} \int_{F_s(T, \chi, \sigma)} [|\eta e^{-A\tau} b_s| - \chi]^2 d\tau = P_s \tag{3.7}$$

where $\mu E_s(T, \chi, \sigma)$ is the Lebesgue measure [11] of the set $E_s(T, \chi, \sigma)$.

The functions $\Phi_1(\chi, \sigma)$ and $\Phi_2(\chi, \sigma)$ are continuous. We consider these functions only in the first quadrant ($\chi \geqslant 0, \sigma \geqslant 0$) of the ($\chi, \sigma$)-plane. As $\chi \to 0$ and $\sigma \to 0$ we have $\Phi_1(\chi, \sigma) \to M_s T$, $\Phi_2(\chi, \sigma) \to M_s^2 T$. The inequalities $M_s T > N_s$ and $M_s^2 T > P_s$ hold for values of T larger than a certain value. For each fixed value of σ the functions $\Phi_1(\chi, \sigma)$ and $\Phi_2(\chi, \sigma)$ decrease strictly monotonically as χ varies from zero to the value

$$\chi' = \max_{t \in [0, T]} (\eta e^{-At} b_s) \tag{3.8}$$

When $\chi = \chi'$, obviously, $\Phi_1(\chi, \sigma) \equiv \Phi_2(\chi, \sigma) \equiv 0$. For each fixed value of $\chi < \chi'$ the functions $\Phi_1(\chi, \sigma)$ and $\Phi_2(\chi, \sigma)$ decrease monotonically as σ increases.

As $\sigma \to \infty$ we have $\Phi_1(\chi, \sigma) \to 0$.

From all that we have said above it follows that Eqs. (3.6) and (3.7) define, in the first quadrant of the (χ, σ)-plane, curves whose endpoints lie on the axes $\chi = 0$ and $\sigma = 0$. Each of these curves is continuous, has only one branch, and does not intersect itself. Each of these curves is monotonic, i.e., if two points $(\chi^{(1)}, \sigma^{(1)})$ and $(\chi^{(2)}, \sigma^{(2)})$ on the curve are such that $\chi^{(2)} > \chi^{(1)}$, then $\sigma^{(2)} < \sigma^{(1)}$. Let us ascertain the relative locations of the points of intersection of curves (3.6) and (3.7) with the axes $\chi = 0$ and $\sigma = 0$. Let the points of intersection of curve (3.6) with the axes $\sigma = 0$ and $\chi = 0$ have the coordinates $(\chi^{(1)}, 0)$ and $(0, \sigma^{(1)})$ respectively, while the points of intersection of curve (3.7) have the coordinates $(\chi^{(2)}, 0)$ and $(0, \sigma^{(2)})$. In other words,

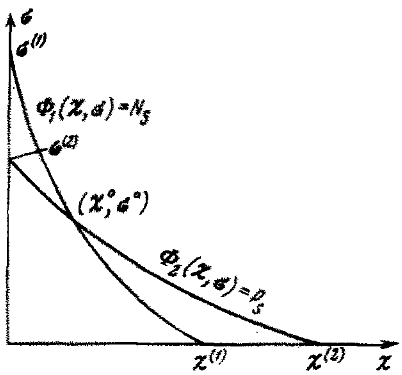

Fig. 2.

$$\Phi_1(\chi^{(1)}, 0) = N_s, \quad \Phi_1(0, \sigma^{(1)}) = N_s$$

$$\Phi_2(\chi^{(2)}, 0) = P_s, \quad \Phi_2(0, \sigma^{(2)}) = P_s$$

As follows from [4], the control $u_s(t, \chi^{(1)}, 0)$ solves the problem of maximizing integral (3.1) under constraints (1.2) and (1.7); consequently, $\Phi_2(\chi^{(1)}, 0) = M_s N_s$. In accordance with condition (3.2), $\Phi_2(\chi^{(1)}, 0) > P_s$. From the monotonicity of the function $\Phi_2(\chi, 0)$ it follows that $\chi^{(1)} < \chi^{(2)}$. As follows from [6, 12], the control $u_s(t, 0, \sigma^{(2)})$ solves the problem of maximizing integral (3.1) under constraints (1.2) and (1.8). In Case A this control does not satisfy condition (1.7), i.e. $\Phi_1(0, \sigma^{(2)}) > N_s$. From the monotonicity of the function $\Phi_1(0, \sigma)$ it follows that $\sigma^{(1)} > \sigma^{(2)}$. Hence we conclude (Fig. 2) that a point of intersection $(\chi^\circ, \sigma^\circ)$ of curves (3.6) and (3.7) exists in the first quadrant. Consequently, the system of Eqs. (3.6) and (3.7) has the solution $\chi^\circ > 0$, $\sigma^\circ > 0$.

It is easy to show (for example, in the same way as in [12]) that the control $u_s(t, \chi^\circ, \sigma^\circ)$ maximizes integral (3.1) under conditions (1.2), (1.7), (1.8). Thus, the maximizing control has been found in Case A; for the distance $d_\eta(T)$ we obtain the expression

$$d_\eta(T) = M_s \int\limits_{E_s(T, \chi^\circ, \sigma^\circ)} |\eta e^{-A\tau} b_s| \, d\tau + \frac{1}{\sigma^\circ} \int\limits_{F_s(T, \chi^\circ, \sigma^\circ)} |\eta e^{-A\tau} b_s| (|\eta e^{-A\tau} b_s| - \chi^\circ) \, d\tau \quad (3.9)$$

The following relations result from expressions (3.5) and (3.6):

$$d_\eta(T) \geqslant M_s \chi^\circ \mu E_s(T, \chi, \sigma) + \frac{\chi^\circ}{\sigma^\circ} \int\limits_{F_s(T, \chi^\circ, \sigma^\circ)} [|\eta e^{-A\tau} b_s| - \chi^\circ] \, d\tau = \chi^\circ N_s \quad (3.10)$$

From expressions (3.5) and (3.7) we obtain

$$d_\eta(T) \geqslant M_s^2\sigma^\circ \mu E_s(T, \chi^\circ, \sigma^\circ) + \frac{1}{\sigma^\circ} \int\limits_{F_s(T, \chi^\circ, \sigma^\circ)} [|\eta e^{-A\tau} b_s| - \chi^\circ]^2 d\tau = \sigma^\circ P_s \quad (3.11)$$

The inequalities

$$\max(\chi^\circ N_s, \ \sigma^\circ P_s) \leqslant d_\eta(T) \leqslant \chi' N_s \tag{3.12}$$

follow from relations (3.10), (3.11), and also from the expressions (1.7), (3.1) and (3.8). The first inequality in (3.12) holds, obviously, for both the control Classes $\Omega_s^m(T)$ $(m = 1, 2)$.

Note that expression (3.4) makes it possible to predetermine the structure of the control, constrained by conditions (1.2), (1.3), (1.4), bringing system (1.1) to the origin in the shortest possible time.

Now let Case B obtain. The maximizing control is determined by the expression $u_s(t, 0, \sigma^{(2)})$. The distance $d_\eta(T)$ is obtained [6] from formula (3.9) if in it we set $\chi^\circ = 0$, $\sigma^\circ = \sigma^{(2)}$.

4. Structure of the controllability regions Q_s^1 and Q^1. Let the roots $\lambda_k = \varepsilon_k + i\omega_k$ with multiplicities p_k of the characteristic equation

$$\det \| A - \lambda E \| = 0 \tag{4.1}$$

have positive real parts for $k = 1, \ldots, r_1$, z ro real parts for $k = r_1 + 1, \ldots, r_2$ and negative real parts for $k = r_2 + 1, \ldots, r_3$. As follows, for example, from [13, 14], the matrix e^{-At} has the form

$$e^{-At} = \sum_{k=1}^{r_3} \sum_{l=0}^{p_k-1} \alpha_{kl} e^{-\lambda_k t} t^l$$

where α_{kl} are constant matrices with the elements α_{kl}^{ij}. The expression for $\eta e^{-At} b_s$ has the form

$$\eta e^{-At} b_s = \sum_{k=1}^{r_3} \sum_{l=0}^{p_k-1} \eta \alpha_{kl} b_s e^{-\lambda_k t} t^l \tag{4.2}$$

Consider the system of linear algebraic equations (in the components of vector η)

$$\eta \alpha_{kl} b_s = 0 \qquad (l = 1, \ldots, p_k - 1 \text{ for } k = r_1 + 1, \ldots, r_2)$$
$$(l = 0, 1 \ldots, p_k - 1 \text{ for } k = r_2 + 1, \ldots, r_3) \tag{4.3}$$

This system consists of

$$\beta = \sum_{k=r_1+1}^{r_3} p_k - (r_2 - r_1)$$

equations. To Eqs. (4.3) we add on the norming condition

$$\sum_{i=1}^{n} \eta_i^2 = 1 \tag{4.4}$$

The vectors η which are the solutions of Eqs. (4.3), (4.4) (we denote them η_s°), and only they, when substituted in (4.2) annihilate all the terms containing $e^{-\varepsilon_k t}$ $(k = r_2 + 1, \ldots, r_3)$, where $\varepsilon_k < 0$, and the terms not containing exponents but containing t^l, where $l \geqslant 1$. Thus the function $|\eta_s^\circ e^{-At} b_s|$ remains bounded as $t \to \infty$. Consequently, as $T \to \infty$ the quantity χ' in (3.8) tends to a finite limit. Since $d_\eta(T)$ is a nondecreasing function of T, it follows from the right-hand inequality in (3.12) that $d_{\eta_s^\circ}(T)$ tends to a finite limit as $T \to \infty$ which we denote $d_{\eta_s^\circ}$ $(d_{\eta_s^\circ} = d_{-\eta_s^\circ})$. If η_s is a vector such that $d_{\eta_s^\circ} \neq 0$, then the set Q_s^1 is included, between the planes

$$\eta_s^\circ x = d_{\eta_s^\circ}, \qquad -\eta_s^\circ x = d_{\eta_s^\circ} \tag{4.5}$$

For those vectors η_s° for which $d_{\eta_s^\circ}(T) < d_{\eta_s^\circ}$ for any finite T, the set $Q_s^1(T)$ reaches the planes (4.5) only as $T \to \infty$ and the coordinates of the points $x \in Q_s^1$ satisfy the strict inequality

$$|\eta_s^\circ x| < d_{\eta_s^\circ} \tag{4.6}$$

If η_s^0 is, for example, a vector such that $\eta_s^\circ e^{-At} b_s \equiv \text{const} \neq 0$, then, there exists a value T' for which $d_{\eta_s^\circ}(T) = d_{\eta_s^\circ}(T') = d_{\eta_s^\circ}$ for all $T \geqslant T'$. For such a vector η_s^0 the set $Q_s^1(T)$ reaches planes (4.5) at $T = T'$ and, for all subsequent increases of T, does not "expand" any more in the direction of η_s°. Consequently, for this vector η_s° on the planes (4.5) there exist points belonging to set Q_s^1. From what has been said we conclude that there exist vectors η_s such that the coordinates of the points of set Q_s^1 satisfy the inequality

$$|\eta_s^\circ x| \leqslant d_{\eta_s^\circ} \tag{4.7}$$

For those and only those vectors η_s° satisfying the system of n algebraic equations

$$\eta x_{kl} b_s = 0 \qquad (k = 1, \ldots, r_3;\ l = 0, 1, \ldots, p_k - 1) \tag{4.8}$$

it is obvious that $\eta_s^\circ e^{-At} b_s \equiv 0$ and $d_{\eta_s^\circ}(T) = d_{\eta_s^\circ} = 0$. System (4.3) is a special case of system (4.8). Let ρ_s be the rank of system (4.8), then the fundamental system of solutions of Eqs. (4.8) consists of $n - \rho_s$ vectors. Let us normalize each of these vectors and denote them $\eta_s^1, \ldots, \eta_s^{n-\rho_s}$. Then the set Q_s^1 belongs to the planes

$$\eta_s^\delta x = 0 \qquad (\delta = 1, \ldots, n - \rho_s) \tag{4.9}$$

i.e., the dimension of set Q_s^1 equals ρ_s. Note that the dimension of set Q_s^1 equals [1, 15] the rank of the matrix $W_s = \| b_s, Ab_s, \ldots, A^{n-1} b_s \|$; when $\rho_s = n$ system (2.2) is completely controllable in Kalman's sense.

Now let $\eta \neq \eta_s^\circ$. We prove that here $d_\eta(T) \to \infty$ as $T \to \infty$. Let us assume at first that Case A holds for all values of T larger than some value. Then, the distance $d_\eta(T)$ satisfies inequality (3.12). We show that the quantity $\max(\chi^\circ, \sigma^\circ)$, being a function of variable T, does not remain bounded as $T \to \infty$. Let us assume the contrary, i.e., let us admit the presence of a constant $c > 0$ such that $\max(\chi^\circ, \sigma^\circ) \leqslant c$ for all values of T. The left-hand side of equality (3.6) can be bound as follows:

$$\Phi_1(\chi^\circ, \sigma^\circ) \geqslant M_s \mu E_s(T, c, c) \tag{4.10}$$

From expression (4.2) follows the relation

$$\left|\eta e^{-At} b_s\right| = e^{-\varepsilon_{k'} t} t^{l'} \left| f_1(t) + f_2(t) \right|$$

where $e^{-\varepsilon_{k'} t} t^{l'}$ is the term having, as $t \to \infty$ the maximal order of growth in comparison with the other terms of form $e^{-\varepsilon_k t} t^l$ occurring in expression (4.2); $f_1(t) \not\equiv 0$ is an almost periodic function, being the sum of a finite number of sinusoids and a constant; $f_2(t) \to 0$ as $t \to \infty$ From the condition $\eta \neq \eta_s$ it follows that $-\varepsilon_{k'} \geqslant 0$ and at least one of the inequalities is fulfilled: $-\varepsilon_{k'} > 0$, $l' > 0$. Consequently, if $\eta \neq \eta_s$, $e^{-\varepsilon_{k'} t} t^{l'} \to \infty$ as $t \to \infty$. As follows from [16], the relation

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f_1(\tau)|\, d\tau = K > 0 \tag{4.11}$$

holds for the almost periodic function $|f_1(t)|$. Using relation (4.11) it is not difficult to show that $\mu E_s(T, c, c) \to \infty$ as $T \to \infty$. From inequality (4.10) we conclude that the left-hand side of relation (3.6) is an unbounded function as $T \to \infty$. Therefore, from the assumption that $\max(\chi^\circ, \sigma^\circ) \leqslant c$ it follows that equality (3.6) cannot hold for sufficiently large values of T, but this contradicts what was presented above. Since the distance $d_\eta(T)$ is a monotonically increasing function of T, then from inequality (3.12) it follows that in Case A. $d_\eta(T) \to \infty$ as $T \to \infty$. If Case B holds for all values of T larger than some value, then, in accordance with [6], $d_\eta(T) \to \infty$ as $T \to \infty$. Let us assume that the intervals of values of T in which Cases A and B hold, alternate. Since the distance $d_\eta(T)$ is a nondecreasing function of T, we can conclude again that $d_\eta(T) \to \infty$ as $T \to \infty$. Thus, the set Q_s^1 is bounded only in the directions of $\eta = \overset{\circ}{\eta}_s$.

The Eqs. (4.3), (4.8), (4.9) and the inequalities (4.6), (4.7) obtained allow us to ascertain completely the structure of the controllability region Q_s^1. Let X^{ρ_s} denote the set of points x satisfying conditions (4.9); if $\rho_s = n$, then $X^{\rho_s} = X$. By $X_1^{\rho_s}$ we denote the subspace of space X^{ρ_s}, spanned by the vectors $\overset{\circ}{\eta}_s$ orthogonal to the vectors η_s^δ $(\delta = 1, \ldots, n - \rho_s)$, and by $X_2^{\rho_s}$ we denote the orthogonal complement of the subspace $X_1^{\rho_s}$ with respect to the space X^{ρ_s}. Thus, the following theorem holds.

Theorem 4.1. The controllability region Q_s^1 is a cylindrical set, i.e., $Q_s^1 = S + X_2^{\rho_s}$, where $S \subset X_1^{\rho_s}$ is a bounded set (the base of the cylinder). When $\rho_s = n$ the dimension of the subspace $X_1^{\rho_s}$ equals the dimension of the fundamental system of solutions of Eqs. (4.3),

$$\sum_{k=1}^{r_1} p_k + (r_2 - r_1)$$

i.e., to the number of eigenvalues of matrix A with positive real parts, with due regard to their multiplicities, and with zero real parts, without regard to their multiplicities. On the boundary of set Q_s^1 there are points both belonging to region Q_s^1 as well as not belonging to it.

Under the condition $\rho_s = n$ we consider two special cases:

1. All roots of Eq. (4.1) have negative real parts. In this case system (4.3) coincides with system (4.8) which, for $\rho_s = n$, has only a trivial solution. Consequently, the quantity $d_\eta(T) \to \infty$ as $T \to \infty$ for all $\eta \neq 0$, and hence, $Q_s^1 = X$.

2. All roots of Eq. (4.1), except λ_1, have negative real parts. The root λ_1 is either a zero root of arbitrary multiplicity p_1 or is a real positive roots of multiplicity $p_1 = 1$.

In this case system (4.3) consists of $n-1$ linearly independent equations. Equations (4.3), (4.4) have only two solutions, differing from each other in sign: η_s° and $-\eta_s^\circ$. The region Q_s^1 is the set of points $x \in X$, bounded by two planes orthogonal to the vector η_s° and located at a distance $d_{\eta_s^\circ}$ from the origin. In the cases when $\varepsilon_1 = 0$ or $\varepsilon_1 > 0$, but N_s and $1/P_s$ are sufficiently small quantities, there are points belonging to set Q_s^1 on the bounding planes.

We now consider the question of the structure of region Q^1. The matrices α_{kl} ($l = 0, 1, \ldots, p_k - 1$) contain p_k linearly independent columns [13, 14], while among the columns of matrices α_{kl} ($l = 1, \ldots, p_k - 1$) there are no more than $p_k - 1$ ones. Therefore, the columns of $\alpha_{kl}b_s$ ($l = 0, 1, \ldots, p_k - 1; s = 1, \ldots, r$) being linear combinations of the columns of matrices α_{kl} ($l = 0, 1, \ldots, p_k - 1$), contain no more than p_k linearly independent, while the columns of $\alpha_{kl}b_s$ ($l = 1, \ldots p_k - 1; s = 1, \ldots, r$) contain no more than $p_k - 1$ ones. Hence it follows that in the system obtained from (4.3) for $s = 1, \ldots, r$, there are no more than β linearly independent ones among the $r\beta$ equations. Let ρ denote the rank of the system of rn equations obtained from (4.8) for $s = 1, \ldots r$. Then, among the vectors η_s^δ ($\delta = 1, \ldots, n - \rho_s; s = 1, \ldots, r$), $n - \rho$ vectors are linearly independent, i.e., the set Q^1 belongs to $n - \rho$ planes of the form (4.9). Note that the dimension of set Q^1 equals the rank of the matrix $\| W_1, \ldots, W_r \|$ [1, 15]; when $\rho = n$ system (1.1) is completely controllable in Kalman's sense.

It is easy to show that when $\rho = n$ in system (4.3) ($s = 1, \ldots, r$) there are no less than $\sum_{k=r_1}^{r_2} p_k$ linearly independent equations (recall that when $\rho_s = n$ we can assert that there are precisely β linearly independent equations in system (4.3)). By X^ρ we denote the set of points x satisfying conditions (4.9) for all $s = 1, \ldots, r$. By X_1^ρ we denote the subspace of space X^ρ. spanned by the vectors η_s^σ ($s = 1, \ldots, r$) orthogonal to the vectors η_s^δ ($\delta = 1, \ldots, n - \rho_s; s = 1, \ldots, r$), and by X_2^ρ we denote the orthogonal complement of subspace X_1^ρ with respect to space X^ρ. The following theorem holds.

Theorem 4.2. The controllability region Q^1 is a cylindrical set, i.e., $Q^1 = S + X_2^\rho$, where $S \subset X_1^\rho$ is a bounded set (the base of the cylinder). When $\rho = n$ the dimension of subspace X_1^ρ, equal to the dimension of the fundamental system of solutions of Eqs. (4.3) ($s = 1, \ldots r$), is not less than

$$\sum_{k=1}^{r_1} p_k + (r_2 - r_1)$$

and not more than

$$\sum_{k=1}^{r_2} p_k$$

On the boundary of set Q^1 there are points both belonging to region Q^1 as well as not belonging to it.

5. Structure of regions Q_s^2 and Q^2. Since $\Omega_s^2(T) \supset \Omega_s^1(T)$, the distance $d_\eta(T)$ for the control class $\Omega_s^2(T)$ is not less than the corresponding distance for class $\Omega_s^1(T)$. In the preceding section we proved for class $\Omega_s^1(T)$ that $d_\eta(T) \to \infty$ as $T \to \infty$, if $\eta \neq \eta_s^\circ$. Consequently, this fact holds also for the control class $\Omega_s^2(T)$. From the right-hand inequality in (3.12) it follows that for $\eta = \eta_s^\circ$ the distance $\bar{d}_\eta(T)$ remains bounded as $T \to \infty$ both for the class $\Omega_s^1(T)$ as well as for the class $\Omega_s^2(T)$. Thus, the very same theorems hold for the controllability regions Q_s^2 and Q^2 as do for the regions Q_s^1 and Q^1. As follows from Theorem 4.1 and from [2, 4], the structure of the controllability region Q_s^m $(m = 1, 2)$ in the cases being considered here coincides with the structure of the controllability region when the control's linear momentum (1.3) is bounded or when the control's magnitude (1.2) and linear momentum (1.3) are bounded simultaneously.

The inference on the structure of the controllability region Q_s^2 could, of course, have been made directly, by solving the problem of maximizing integral (3.1) in the control class $\Omega_s^2(T)$. Here the control $u_s(t) \in \Omega_s^2(T)$ maximizing integral (3.3) has the form

$$u_s(t, \chi, \sigma) = \begin{cases} \sigma^{-1}[|\eta e^{-At} b_s| - \chi], & t \in F_s(T, \chi, \infty) = E_s(T, \chi, 0) \\ 0, & t \in G_s(T, \chi) \end{cases}$$

In the case analogous to Case A the equations determining the values χ° and σ° acquire the form

$$\frac{1}{\sigma} \int\limits_{E_s(T, \chi, 0)} (|\eta e^{-A\tau} b_s| - \chi)\, d\tau = N_s, \qquad \frac{1}{\sigma^2} \int\limits_{E_s(T, \chi, 0)} [|\eta e^{-A\tau} b_s| - \chi]^2\, d\tau = P_s$$

In this case the distance $d_\eta(T)$ is determined by the formula

$$d_\eta(T) = \frac{1}{\sigma^\circ} \int\limits_{E_s(T, \chi^\circ, 0)} |\eta e^{-A\tau} b_s| (|\eta e^{-A\tau} b_s| - \chi^\circ)\, d\tau$$

6. Example. Consider the system of equations

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = a_{22}x_2 + a_{23}x_3 + b_2 u, \quad x_3^{\cdot} = a_{32}x_2 + a_{33}x_3 + b_3 u \tag{6.1}$$

Equations (6.1) describe the motion of a winged aircraft in a horizontal plane (x_1 and x_3 are the heading and angle of side slip). The index s is dropped because there is only one control in system (6.1). The characteristic equation of system (6.1) has one zero root; let the other two roots be real, simple (therefore, we drop the index l also), and negative, so that $\lambda_1 = 0$, $\lambda_2 < 0$, $\lambda_3 < 0$. The matrix e^{At} has the form

$$e^{At} = \left\| \begin{matrix} 1 & \alpha_1^{(1,2)} + \alpha_2^{(1,2)} e^{\lambda_2 t} + \alpha_3^{(1,2)} e^{\lambda_3 t} & \alpha_1^{(1,3)} + \alpha_2^{(1,3)} e^{\lambda_2 t} + \alpha_3^{(1,3)} e^{\lambda_3 t} \\ 0 & \alpha_2^{(2,2)} e^{\lambda_2 t} + \alpha_3^{(2,2)} e^{\lambda_3 t} & \alpha_2^{(2,3)} e^{\lambda_2 t} + \alpha_3^{(2,3)} e^{\lambda_3 t} \\ 0 & \alpha_2^{(3,2)} e^{\lambda_2 t} + \alpha_3^{(3,2)} e^{\lambda_3 t} & \alpha_2^{(3,3)} e^{\lambda_2 t} + \alpha_3^{(3,3)} e^{\lambda_3 t} \end{matrix} \right\|$$

where the coefficients $\alpha_k^{(i,j)}$ are expressed in a definite manner [2, 4] in terms of the coefficients of matrix A. The expression $\eta e^{-At} b$ is written in the form

$$\eta e^{-At} b = \eta_1 \alpha_1^{(1)} + e^{-\lambda_2 t}(\eta_1 \alpha_2^{(1)} + \eta_2 \alpha_2^{(2)} + \eta_3 \alpha_2^{(3)}) + e^{-\lambda_3 t}(\eta_1 \alpha_3^{(1)} + \eta_2 \alpha_3^{(2)} + \eta_3 \alpha_3^{(3)})$$

where

$$\alpha_k^{(i)} = b_2 \alpha_k^{(i,2)} + b_3 \alpha_k^{(i,3)} \quad (i = 1 \text{ for } k = 1;\ i = 1, 2, 3 \text{ for } k = 2, 3)$$

Equations (4.3), (4.4) have the form

$$\eta_1\alpha_k^{(1)} + \eta_2\alpha_k^{(2)} + \eta_3\alpha_k^{(3)} = 0 \quad (k = 2, 3), \quad \eta_1^2 + \eta_2^2 + \eta_3^2 = 1 \tag{6.2}$$

If system (6.1) is completely controllable, then Eqs. (6.2) have only two solutions, η° and $-\eta^\circ$, and, moreover,

$$\eta_i = \Delta_i / \Delta \qquad (i = 1,2,3)$$

where $\Delta_1 = \alpha_2^{(2)}\alpha_3^{(3)} - \alpha_3^{(2)}\alpha_2^{(3)}$, while Δ_2 and Δ_3 are obtained by a cyclic permutation of the upper index in the coefficients $\alpha_k^{(i)}$ of the expression for Δ_1; $\Delta^2 = \Delta_1^2 + \Delta_2^2 + \Delta_3^2$. Thus, $\eta^\circ e^{-At} b = \Delta_1\alpha_1^{(1)}/\Delta \equiv \text{const}$. Therefore, for both classes of admissible controls we have

$$d_{\eta^\circ} = N \, | \Delta_1\alpha_1^{(1)} | \, \Delta^{-1}$$

Consequently, the controllability regions Q^m $(m = 1,2)$ are, just as in [2, 4], sets of phase space points bounded by the two planes

$$\Delta_1 x_1 + \Delta_2 x_2 + \Delta_3 x_3 = \pm N\Delta_1\alpha_1^{(1)}$$

On these planes there are points belonging to the regions Q^m $(m = 1,2)$.

BIBLIOGRAPHY

1. Kalman, R. E., On the general theory of control systems. Proc. First Internat. Congress IFAC, Vol. 2. Moscow, Izd. Akad. Nauk SSSR, 1961.
2. Formal'skii, A. M., Construction of the controllability region for systems with controls of constrained linear momentum. Vestn. Moskov. Gos. Univ., Ser. Mat. i Mekh., №5, 1966.
3. Formal'skii, A. M., Determination of the controllability region for linear stationary systems. In: Physics, Mathematics, Mechanics. Proc. First Moscow Conf. Young Scientists. Moscow, "Nauka", 1968.
4. Formal'skii, A. M., Controllability region of systems with restricted control resources. Avtomatika i Telemekhanika, №3, 1968.
5. Formal'skii, A. M., The time-optimal control problem in systems with controlling forces of bounded magnitude and impulse. PMM Vol. 34, №5, 1970.
6. Formal'skii, A. M., Controllability region of systems having controls with constrained magnitude and energy. Vestn. Moskov. Gos. Univ., Ser. Mat. i Mekh., №5, 1966.
7. Liusternik, L. A. and Sobolev, V. I., Elements of Functional Analysis. Moscow, "Nauka", 1965.
8. Bellman, R., Glicksberg, I., and Gross, O., Certain Questions in the Mathematical Theory of Control Processes. Moscow, Izd. Inostr. Lit., 1962.
9. Beckenbach, E. and Bellman, R., Inequalities. Moscow, "Mir", 1965.
10. Eaton, J. H., An iterative solution to time optimal control. J. Math. Anal. and Appl. Vol. 5, №2, 1962.
11. Shilov, G. E., Mathematical Analysis. Moscow, Fizmatgiz, 1960.
12. Gabasov, R. and Gindes, V. B., On optimal processes in linear systems with two constraints on the controls. Avtomatika i Telemekhanika Vol. 26, №6, 1965.
13. Bulgakov, B. V., Oscillations. Moscow, Gostekhizdat, 1954.

14. Gantmakher, F. R., Theory of Matrices. Moscow, "Nauka", 1967.
15. Krasovskii, N. N., Theory of Control of Motion. Moscow, "Nauka", 1968.
16. Demidovich, B. P., Lectures on the Mathematical Theory of Stability. Moscow, "Nauka", 1967.

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CONTACT PROBLEM FOR A SEMI-INFINITE CYLINDRICAL SHELL

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The problem of the impression of pointed stamps along segments of the cross-sectional circle of a semi-infinite cylindrical shell supported freely at the endface is considered. The edges of the stamps are absolutely stiff, of constant radius, and have no sharp angles. The influence of the shell endface on the character of the change in reaction of the stamps is investigated. The problem is solved on the basis of the shell theory equations constructed taking account of the Kirchoff-Love hypothesis. The friction between the shell surface and the stamp edges is not taken into account.

1. Let us consider a semi-infinite cylindrical shell (Fig. 1), freely supported on the endface $\xi = 0$ compressed along segments of the circle $\xi = \xi_0$ by identical stamps, where m denotes the number of stamps ($(m = 2)$ in Fig. 1).

We consider the stamp edges to be sharp and absolutely stiff so that the contact between the shell and stamp is on the arc of a circle whose magnitude is characterized by the central angle θ to be determined. We consider the curvature $1/R_1$ of the stamp edges to be constant. Linear stress resultants q (reactions) act from the stamp on the shell, and we consider them directed along the normal to the surface within the shell, without taking account of friction. Proceeding from the linear theory of thin shallow shells, we shall also assume that either the angle θ is small, or the radius R_1 of the stamp edges differs slightly from the radius of the outer surface R_0 of the shell.

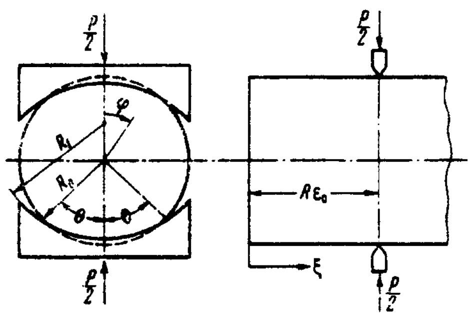

Fig. 1.

We obtain the initial equation of the problem from the condition of complete abutment of the shell to the stamp in the contact zone, which can be written as $\varkappa_2 = 1/R_1 - 1/R_0$, where $\varkappa_2$ is the bending strain of the shell in the circumferential direction on the line of contact. Knowing the Green's function Ψ for a semi-infinite shell freely supported on the endface $\xi = 0$ the strain $\varkappa_2$ can be determined by formulas from [1]. Let us show that